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Learning Markov chains in fractal compression of image data

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Abstract. In a recent paper [17] we proposed a stochastic algorithm which generates optimal probabilities for the decompression of an image represented by the fixed point of an IFS system (SAOP). We show here that such an algorithm is in fact a non trivial example of Generalized Random System with Complete Connections. We also exhibit a generalization which could represent the solution to the inverse problem for an image with grey levels, if a fixed set of contraction maps is available.

Keywords: Image data, stochastic algorithms, random systems with complete connections.

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1 Introduction

This paper is concerned with the problem of compressing image data, by means of Iterated Function Systems or of Iterated Function Systems with Probability (IFS or IFSP systems).

If an image is represented by a compact subset of the unit square, one says that it is exactly compressed by an IFS system (or that it is generated by an IFS system) if there exists a set of maps on the unit square, contractive with respect to the Euclidean metrics, such that the operator obtained by their "superposition", called the Iterated Function System operator, has such a subset as fixed point. Then, the set of points generated by the iterative application of the IFS operator to any initial compact subset of the unit square, asymptotically reproduces the image.

We can also associate a probability to every assigned contraction map and, in place of iteratively applying the IFS operator, we apply, at every step, only one

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map, chosen according to the corresponding probability. The resulting random walk on the unit square and the IFS system with the assigned probabilities are respectively called Chaos-Game and IFSP system.

An IFSP system can in principle generate an image with grey levels, if the frequency of the visits in the various points of the image by the Chaos-Game is roughly interpreted as their grey level.

In case of a black and white image, as it is, in fact, when the image itself is represented by a subset of the unit square, the best choice of the set of probabilities is that by which the frequencies of the visits in the various pixels of the unit square by the Chaos-Game reproduce, as well as possible, the uniform distribution on the subset representing the image.

With such a set of optimal probabilities, a computer simulation of the Chaos-Game generates the image associated to the IFS system in the fastest way, because the possibility of visiting twice a pixel is approximately the same for all the points of the image (it is exactly the same if the maps are not overlapping).

In a recent paper [17], we faced this problem and proposed a stochastic algorithm to compute such optimal probabilities for an arbitrary IFS system on the pixels space (SAOP).

As we will describe in Section 5, SAOP can be extended to the case with grey levels, in the sense that, given a fixed set of contraction maps and an image represented by a grey level function (a function which associates to any point its grey level, with zero for the white and one for the black), the algorithm computes the probabilities to be assigned to the maps, in order to get the best approximation of the above mentioned grey level function, through the frequency of the visits in the pixels by the associated Chaos-Game.

As a consequence, this extension of SAOP represents a solution to the inverse problem of finding the best IFSP system which approximates the image, for a given set of available maps.

The probabilistic description of SAOP is non-trivial and in fact it turns out to be a quite complicated example of Random System with Complete Connections. In particular, the set of probabilities associated to the maps evolves in time according to a Learning Markov Chain.

The plan of the paper is the following: in Section 2 we recall the basic aspects of the IFS and IFSP systems and the construction of SAOP. In Section 3 we mention the definitions concerning the Random Systems with Complete Connections and formalize SAOP as a particular one of them. In Section 4 we report some results on convergence of the algorithm. In Section 5 we briefly discuss the inverse problem and introduce a generalization of SAOP to the case of grey levels.

Finally, in Section 6 some comments about possible developements are given.

2 IFS systems and SAOP algorithm

Let $(\mathcal{X}, d)$ be the complete metric space representing the "base space": we identify it with the finite pixels space, i.e. a finite subset of $\mathbb{Z}^2$, with the Euclidean metrics; let $\mathbf{w} = (w_1, w_2, \dots, w_N)$ denote the set of discretized contraction maps which describe the image, defined on $\mathcal{X}$: the pair $(\mathcal{X}, \mathbf{w})$ is called an IFS (Iterated Fuction System).

We associate with the IFS system a set of probabilities $\mathbf{p} = (p_1, \dots, p_N)$, $p_i > 0$, $\sum_{i=1}^N p_i = 1$, where p_i represents the probability for the map w_i to be chosen. The new system $(\mathcal{X}, \mathbf{w}, \mathbf{p})$ is called an IFSP (Iterated Function System with Probability).

Let A be the attractor of the system, i.e. the image we want to reconstruct, compressed by means of the maps $(w_1, w_2, \dots, w_N)$; $A \in K(\mathcal{X})$, where $(K(\mathcal{X}), h)$ is the complete metric space of compact subsets of $\mathcal{X}$, with the Hausdorff metrics. Since the maps $(w_1, w_2, \dots, w_N)$ are contractive, then A is the unique fixed point of the IFS operator $W(\cdot) = \bigcup_{i=1}^N w_i(\cdot)$ defined on $K(\mathcal{X})$ (see [2] and [3]).

In order to decompress it, the algorithm which is tipically used is the so-called Chaos Game, that, roughly speaking, can be described by the following steps:

- fix a starting point $x_0 \in \mathcal{X}$;
- choose a map w_i with probability p_i;
- apply the map w_i to x_0, obtaining a new point $x_1 = w_i(x_0) \in \mathcal{X}$;
- choose a new map, independently with respect to the first;
- iterate n times and get the sequence of points $x_0, \dots, x_n$: the set $\{x_0, \dots, x_n\}$ approximates A.

Formally, given an appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$, for example the canonical trajectory space, we define the Chaos-Game as a Random Walk $(X_n)_{n=0}^\infty$ on $\mathcal{X}$: let $(Z_n)_{n=0}^\infty$ be i.i.d. random variables, such that

$$\mathbb{P}[Z_n = i] = p_i, \ \forall i = 1 \dots N,$$

then, assuming that the initial value is randomly chosen, the process $(X_n)_{n=0}^\infty$ is defined by

$$\begin{cases} X_0 \sim U(\mathcal{X}), \\ X_{n+1} = w_{Z_n}(X_n), \ \forall n \geq 0, \end{cases}$$

where $U(\mathcal{X})$ is the uniform distribution on $\mathcal{X}$.

$(X_n)_{n=0}^{\infty}$ is of course an homogeneous Markov chain with state space $\mathcal{X}$ and its transition matrix $\mathbf{P}$, whose elements are defined by

$$p_{ij} := \mathbb{P}[X_{n+1} = x_j | X_n = x_i] = \sum_{k: w_k(x_i) = x_j} p_k, \quad \forall x_i, x_j \in \mathcal{X}.$$

Under our assumptions, by Elton's theorem [7],

$$\lim_{m \uparrow \infty} \frac{1}{m} \sum_{n=1}^{m} \delta_{X_n, x} = \pi_x, \quad \text{a.s. } \forall x \in \mathcal{X},$$

where $(\pi_x)_{x \in \mathcal{X}}$ is the only invariant probability measure for the transition matrix $\mathbf{P}$. Indeed one could easily see that A is a closed recurrent class for $(X_n)_{n=0}^{\infty}$, and that $\mathcal{X} \setminus A$ contains only transient states (see [16]). The result follows then by standard results of Markov chains theory.

SAOP (Stochastic Algorithm for the Optimization of Probabilities) decompresses the image A, optimizing, in the meanwhile, the probabilities associated to the single maps.

Its basic idea consists in using the Chaos-Game, with a fixed set of probabilities, which is improved whenever an error occurs, in the following way:

- choose an initial distribution $\mathbf{p}^0 = (p_1^0, \ldots, p_N^0)$;
- start with the Chaos-Game;
- stop when a site already visited is reached for the second time;
- give a penalty to the map which made the mistake, reducing its probability;
- consider a new distribution $\mathbf{p}$ in which the map is penalized, and one of the others is randomly rewarded;
- use the Chaos-Game again and iterate.

Formally, we have a process on $\mathcal{X}$, $(\tilde{X}_m)_{m=1}^{\infty}$, the Time Dependent Chaos-Game (TDCG), consisting of a sequence of different Chaos-Games.

In the following, we will assume all random variables defined on a suitable fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We now give a mathematical description of SAOP, following [17], but with slightly different notations, which will be useful later to prove that it satisfies the assumptions characterizing a Random System with Complete Connections. We will proceed by induction.

Initial step: the initial probabilities $\mathbf{p}^0$ & Block 1. Let us fix an initial set of probabilities $\mathbf{p}^0 = (p_1^0, \ldots, p_N^0)$, for example $p_i^0 = \frac{1}{N}$, $\forall i = 1 \ldots N$, and consider $(Z_n^0)_{n=1}^{\infty}$, i.i.d. random variables such that

$$\mathbb{P}[Z_n^0 = i] = p_i^0, \ \forall i = 1 \ldots N.$$

The corresponding Chaos-Game is defined by the process $(X_n^1)_{n=1}^{\infty}$, where:

$$\begin{cases} X_1^1 \sim U(\mathcal{X}), \\ X_{n+1}^1 = w_{Z_n^0}(X_n^1), \ \forall n \geq 1, \end{cases}$$

and $U(\mathcal{X})$ is the uniform distribution on $\mathcal{X}$.

The length of Block 1 is represented by the stopping time

$$\lambda_1 = \inf\{h \geq 2 : \exists 1 \leq k < h \text{ s.t. } X_k^1 = X_h^1\}.$$

Inductive step: the probabilities $\mathbf{p}^j$ & Block $j+1$. Suppose that probabilities $\mathbf{p}^{j-1}$ and Block j are already defined and let

$$\lambda_j = \inf\{h \geq 2 : \exists 1 \leq k < h \text{ s.t. } X_k^j = X_h^j\}, \tag{1}$$

be the length of Block j.

Let us also consider the random variable ζ_j, denoting the map chosen to be rewarded in Block $j+1$:

$$\mathbb{P}[\zeta_j = i] = \begin{cases} 0 & \text{if } Z_{\lambda_j - 1}^{j-1} = i, \\ \frac{1}{N-1} & \text{otherwise.} \end{cases} \tag{2}$$

We can then consider the new set of probabilities $\mathbf{p}^j = (p_0^j, \ldots, p_N^j)$ where, with fixed $\Delta \in (0, 1)$,

$$p_i^j = \begin{cases} p_i^{j-1} - \Delta & \text{if} \quad Z_{\lambda_j-1}^{j-1} = i, \\ p_i^{j-1} & \text{if} \quad Z_{\lambda_j-1}^{j-1} \neq i, \zeta_j \neq i, \\ p_i^{j-1} + \Delta & \text{if} \quad Z_{\lambda_j-1}^{j-1} \neq i, \zeta_j = i. \end{cases} \tag{3}$$

If the map which failed is w_i, its probability is reduced of the fixed quantity Δ, while if another one made the mistake, its probability is increased of Δ if w_i is chosen to be rewarded, it is unchanged otherwise.

Let us consider also $(Z_n^j)_{n=1}^\infty$ i.i.d. random variables such that

$$\mathbb{P}[Z_n^j = i] = p_i^j, \ \forall i = 1 \dots N; \tag{4}$$

the Chaos-Game is now $(X_n^{j+1})_{n=1}^\infty$, where:

$$\begin{cases} X_1^{j+1} \sim U(\mathcal{X}), \\ X_{n+1}^{j+1} = w_{Z_n^j}(X_n^{j+1}), \ \forall n \geq 1. \end{cases} \tag{5}$$

The length of Block $j+1$ is then given by

$$\lambda_{j+1} = \inf\{h \geq 2 : \exists 1 \leq k < h \text{ s.t. } X_k^{j+1} = X_h^{j+1}\}. \tag{6}$$

The TDCG $(\tilde{X}_m)_{m=1}^\infty$, is then defined by:

$$\tilde{X}_m = X^j_{m-T_{j-1}}, \ \text{if } m \in \{T_{j-1}+1, \dots, T_j\},$$

where

$$\begin{cases} T_0 = 0, \\ T_j = \sum_{i=1}^j \lambda_i, \ \forall j \geq 1. \end{cases}$$

TDCG is a sort of regenerative time-dependent process and it is a Markov chain (a Random Walk on $\mathcal{X}$), for a fixed choice of the sequences $(\mathbf{p}^n)_{n=0}^\infty$ and $(\lambda_n)_{n=1}^\infty$.

The process which describes the corresponding evolution of the probabilities is

$$\mathbf{Y}_j = \mathbf{p}^j, \ \forall j \geq 0, \tag{7}$$

where its value at step j is the set of probabilities used in the stochastic interval $[T_j+1, T_{j+1}]$ (for more details, see [17]).

3 SAOP as a Random System with Complete Connections

In this section we will show how SAOP can be seen as a non-trivial example of Random System with Complete Connections.

First of all, we recall the basic notions of the theory of Dependence with Complete Connections, exhaustively surveyed in [12], by Iosifescu and Grigorescu.

The mentioned theory was introduced in 1935, by Onicescu and Mihoc [19], and studied afterwards by the Romanian school, with Ciucu, Theodorescu, Iosifescu and others (see, for example, [5], [13]). It is a non-trivial extension of Markovian Dependence theory, and it was also investigated by Doeblin and Fortet [6] and by Harris [11].

Examples of Random Systems with Complete Connections are stochastic learning models, urn models, partially observed random chains, decision models and others.

Definition 3.1. *An homogeneous* **Random System with Complete Connections** *or* **RSCC** *is a quadruple* $\{(V, \mathcal{V}), (H, \mathcal{H}), u, P\}$ *where*

(i) $(V, \mathcal{V})$ *and* $(H, \mathcal{H})$ *are arbitrary measurable spaces;*

(ii) $u : V \times H \to V$ *is a* $(\mathcal{V} \otimes \mathcal{H}, \mathcal{V})$*-measurable map;*

(iii) P *is a transition probability function from* $(V, \mathcal{V})$ *to* $(H, \mathcal{H})$*, i.e. a real valued function defined on* $V \times \mathcal{H}$*, such that* $P(v, \cdot)$ *is a probability on* $(H, \mathcal{H})$ *for any* $v \in V$*, and* $P(\cdot, A)$ *is a random variable on* $(V, \mathcal{V})$ *for any* $A \in \mathcal{H}$*.*

A generalization of this definition is due to Le Calvé and Theodorescu [14]:

Definition 3.2. *An homogeneous* **Generalized Random System with Complete Connections** *or* **GRSCC** *is a quadruple* $\{(V, \mathcal{V}), (H, \mathcal{H}), \Pi, P\}$ *where*

(i) $(V, \mathcal{V})$ *and* $(H, \mathcal{H})$ *are arbitrary measurable spaces;*

(ii) Π *is a transition probability function from* $(V \times H, \mathcal{V} \otimes \mathcal{H})$ *to* $(V, \mathcal{V})$*;*

(iii) P *is a transition probability function from* $(V, \mathcal{V})$ *to* $(H, \mathcal{H})$*.*

In both cases an existence theorem was proved: we state here only the one concerning the GRSCC.

Theorem 3.3. *For a given homogeneous GRSCC and an arbitrarily fixed* $v_0 \in V$*, there exist a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, a sequence* $(\xi_n)_{n=1}^{\infty}$ *of* H*-valued random variables and a sequence* $(\eta_n)_{n=0}^{\infty}$ *of* V*-valued random variables, both defined on* Ω*, such that:*

(i) for any $A \in \mathcal{H}$*,* $B \in \mathcal{V}$ *and* $n \geq 1$ *we have,*

$$\begin{aligned}
&\mathbb{P}[\eta_0 \in B] = \delta_{v_0}(B),\\
&\mathbb{P}[\xi_1 \in A] = P(v_0, A),\\
&\mathbb{P}[\xi_{n+1} \in A | \xi_1, \dots, \xi_n, \eta_0, \dots, \eta_n] = P(\eta_n, A),\ \mathbb{P} - a.s.,\\
&\mathbb{P}[\eta_{n+1} \in B | \xi_1, \dots, \xi_{n+1}, \eta_0, \dots, \eta_n] = \Pi(\eta_n, \xi_{n+1}, B),\ \mathbb{P} - a.s.;
\end{aligned}$$

(ii) the sequence $(\eta_n)_{n=0}^{\infty}$ *is an homogeneous Markov Chain, whose transition probability function* Q *and infinitesimal generator* L *are given by the equations*

$$Q(v, B) = \int_H \Pi(v, x, B) P(v, dx), \tag{8}$$

for all $v \in V$ *and* $B \in \mathcal{V}$, *and*

$$Lf(v) = \int_V f(v') Q(v, dv') = \int_V f(v') \int_H \Pi(v, x, dv') P(v, dx), \tag{9}$$

for all f *bounded* $\mathcal{V}$*-measurable functions on* V.

The sequence $(\xi_n)_{n=1}^{\infty}$ is called the **Generalized Chain with Complete Connections (GCCC)** or **Generalized Chain of Infinite Order** (this second term was coined by Harris [11]), while $(\eta_n)_{n=0}^{\infty}$ is the **Markov Chain associated to the GRSCC**. Notice that while η_{n+1} depends both on η_n and ξ_{n+1}, the law of ξ_{n+1} depends only on η_n.

We shall see that SAOP is an example of GRSCC, where, roughly speaking, the sequence of Blocks is a GCCC and the sequence of probabilities is the associated Markov Chain.

We first particularize the two measurable spaces, introduced in Definition 3.2: $V = \{\mathbf{p} \in E^N : \sum_{i=1}^N p_i = 1\}$, where $E = \{0, \Delta, \dots, (M-1)\Delta, 1\}$, $\Delta = \frac{1}{M}$, $M \in \mathbb{N}$; $\mathcal{V} = \mathcal{P}(V)$; and $H = \mathcal{X}^{N_{\mathcal{X}}+1}$, where $N_{\mathcal{X}}$ is the number of pixels of $\mathcal{X}$; $\mathcal{H} = \mathcal{P}(H)$.

The two chains, $(\eta_n)_{n=0}^{\infty}$ and $(\xi_n)_{n=1}^{\infty}$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values on the above defined spaces, are respectively:

$$\begin{aligned}\eta_n &:= \mathbf{Y}_n \\ \xi_n &:= \mathbf{X}_n := (X_1^n, \dots, X_{\lambda_n}^n, X_{\lambda_n+1}^n, \dots, X_{N_{\mathcal{X}}+1}^n).\end{aligned}$$

Moreover the initial condition $\mathbf{p}^0 \in V$ for $(\eta_n)_{n=0}^{\infty}$, is given.

Notice that ξ_n is the realization of the n^{th} Chaos-Game, i.e. the Chaos-Game which is ruled by the $(n-1)^{th}$ set of probabilities $\mathbf{p}^{n-1}$, in the time interval $[1, N_{\mathcal{X}} + 1]$. Of course, it is not necessary to consider the Chaos-Game as indefinitely evolving, because the n^{th} Block stops at time $\lambda_n \le N_{\mathcal{X}} + 1$.

Observe that, by construction, both

$$\begin{aligned}&\mathbb{P}[\mathbf{X}_{n+1} \in A | \mathbf{Y}_n, \dots, \mathbf{Y}_0, \mathbf{X}_n, \dots, \mathbf{X}_1] \\ &\quad = \mathbb{P}[\mathbf{X}_{n+1} \in A | \mathbf{Y}_n] := P(\mathbf{Y}_n, A), \ \forall A \in \mathcal{H}\end{aligned} \tag{10}$$

and

$$\begin{aligned}&\mathbb{P}[\mathbf{Y}_{n+1} \in B|\mathbf{Y}_n, \dots, \mathbf{Y}_0, \mathbf{X}_{n+1}, \dots, \mathbf{X}_1] \\ &= \mathbb{P}[\mathbf{Y}_{n+1} \in B|\mathbf{Y}_n, \mathbf{X}_{n+1}] := \Pi(\mathbf{Y}_n, \mathbf{X}_{n+1}, B), \ \forall B \in \mathcal{V} \qquad (11)\end{aligned}$$

hold (i.e. *(i)* of Theorem 3.3).

In order to define completely the GRSCC we finally give the expression of the transition probability functions Π and P.

To define $P(\mathbf{p}, \mathbf{x})$, $\forall \mathbf{p} \in V$, $\mathbf{x} \in H$, let us observe that the sets

$$A_{m,l} := \{\mathbf{x} \in H : x_i \neq x_j, \ \forall i, j = 1 \dots l-1, x_l = x_m\},$$

$\forall 1 \leq m < l \leq N_{\mathcal{X}} + 1$, form a partition of H.

Then, using the independence of $(Z_k^0)_{k=1}^{\infty}$, we get, $\forall \mathbf{x}^{m,l} \in A_{m,l}$, $1 \leq m < l \leq N_{\mathcal{X}} + 1$,

$$\begin{aligned} P(\mathbf{p}, \mathbf{x}^{m,l}) &= \mathbb{P}[\mathbf{X}_{n+1} = \mathbf{x}^{m,l}|\mathbf{Y}_n = \mathbf{p}] = \mathbb{P}[\mathbf{X}_1 = \mathbf{x}^{m,l}|\mathbf{Y}_0 = \mathbf{p}] \\ &= \mathbb{P}[X_1^1 = x_1^{m,l}] \cdot \mathbb{P}[X_2^1 = x_2^{m,l}|X_1^1 = x_1^{m,l}, \mathbf{Y}_0 = \mathbf{p}] \cdots \\ &\qquad \cdots \mathbb{P}[X_{N_{\mathcal{X}}+1}^1 = x_{N_{\mathcal{X}}+1}^{m,l}|X_{N_{\mathcal{X}}}^1 = x_{N_{\mathcal{X}}}^{m,l}, \mathbf{Y}_0 = \mathbf{p}] \\ &= \frac{1}{N_{\mathcal{X}}} \cdot \prod_{h=1}^{N_{\mathcal{X}}} \left(\sum_{k: w_k(x_h^{m,l}) = x_{h+1}^{m,l}} p_k \right) := \alpha(\mathbf{x}^{m,l}, \mathbf{p}). \qquad (12) \end{aligned}$$

Let us now construct $\Pi(\mathbf{p}, \mathbf{x}^{m,l}, \mathbf{q})$, $\forall \mathbf{p}, \mathbf{q} \in V$, $\mathbf{x}^{m,l} \in A_{m,l}$, $1 \leq m < l \leq N_{\mathcal{X}} + 1$. Let us observe that

$$\Pi(\mathbf{p}, \mathbf{x}^{m,l}, \mathbf{q}) = \mathbb{P}[\mathbf{Y}_{n+1} = \mathbf{q}|\mathbf{X}_{n+1} = \mathbf{x}^{m,l}, \mathbf{Y}_n = \mathbf{p}] \neq 0$$

only if $\mathbf{q} = \mathbf{p}(i, j)$, for some $i \neq j \in \{1 \dots N\}$, where

$$p_k(i, j) = \begin{cases} p_k & \text{if } k \neq i, j \\ p_i - \Delta & \text{if } k = i \\ p_j + \Delta & \text{if } k = j. \end{cases}$$

Then

$$\begin{aligned} &\Pi(\mathbf{p}, \mathbf{x}^{m,l}, \mathbf{p}(i, j)) = \mathbb{P}[\mathbf{Y}_{n+1} = \mathbf{p}(i, j)|\mathbf{X}_{n+1} = \mathbf{x}^{m,l}, \mathbf{Y}_n = \mathbf{p}] \\ &= \mathbb{P}[\mathbf{Y}_1 = \mathbf{p}(i, j)|\mathbf{X}_1 = \mathbf{x}^{m,l}, \mathbf{Y}_0 = \mathbf{p}] \\ &= \mathbb{P}[Z_{\lambda_1 - 1}^0 = i, \zeta_1 = j|\mathbf{X}_1 = \mathbf{x}^{m,l}, \mathbf{Y}_0 = \mathbf{p}] \\ &= \mathbb{P}[\zeta_1 = j|Z_{\lambda_1 - 1}^0 = i] \cdot \mathbb{P}[Z_{l-1}^0 = i|\mathbf{X}_1 = \mathbf{x}^{m,l}, \mathbf{Y}_0 = \mathbf{p}] \\ &= \frac{1}{N-1} \cdot \frac{\mathbb{P}[Z_{l-1}^0 = i, \mathbf{X}_1 = \mathbf{x}^{m,l}|\mathbf{Y}_0 = \mathbf{p}]}{\mathbb{P}[\mathbf{X}_1 = \mathbf{x}^{m,l}|\mathbf{Y}_0 = \mathbf{p}]} = \frac{1}{N-1} \cdot \frac{a^i(\mathbf{x}^{m,l}, \mathbf{p})}{\alpha(\mathbf{x}^{m,l}, \mathbf{p})} \end{aligned}$$

where

$$\begin{aligned} a^i(\mathbf{x}^{m,l},\mathbf{p}) := & \ \mathbb{P}[Z^0_{l-1}=i,\mathbf{X}_1=\mathbf{x}^{m,l}|\mathbf{Y}_0=\mathbf{p}] \\ = & \ \frac{1}{N_{\mathcal{X}}}\cdot\prod_{h\neq l-1}\left(\sum_{k:w_k(x_h^{m,l})=x_{h+1}^{m,l}} p_k\right) p_i \mathbf{1}_{\left\{w_i(x_{l-1}^{m,l})=x_l^{m,l}\right\}}. \end{aligned} \tag{13}$$

Notice that

$$\alpha(\mathbf{x}^{m,l},\mathbf{p})=\sum_{i=1}^{N} a^i(\mathbf{x}^{m,l},\mathbf{p}). \tag{14}$$

So far we have proved that

$$\Pi(\mathbf{p},\mathbf{x}^{m,l},\mathbf{q})=\begin{cases} \frac{1}{N-1}\cdot\frac{a^i(\mathbf{x}^{m,l},\mathbf{p})}{\alpha(\mathbf{x}^{m,l},\mathbf{p})}, & \text{if } \mathbf{q}=\mathbf{p}(i,j) \\ & \text{for some } i\neq j=1\ldots N, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

Remark 3.4. By condition *(ii)* of Theorem 3.3, $(\mathbf{Y}_n)_{n=0}^{\infty}$ is a Markov Chain and its transitions can be immediately computed: if $\mathbf{q}\neq\mathbf{p}(i,j)\ \forall i\neq j=1\ldots N$, then $Q(\mathbf{p},\mathbf{q})=0$, otherwise

$$\begin{aligned} Q(\mathbf{p},\mathbf{q}) &= \sum_{\mathbf{x}\in H}\Pi(\mathbf{p},\mathbf{x},\mathbf{q})\cdot P(\mathbf{p},\mathbf{x}) \\ &= \sum_{\mathbf{x}\in H}\frac{1}{N-1}\cdot\frac{a^i(\mathbf{x},\mathbf{p})}{\alpha(\mathbf{x},\mathbf{p})}\cdot\alpha(\mathbf{x},\mathbf{p}) = \frac{1}{N-1}\cdot\sum_{\mathbf{x}\in H} a^i(\mathbf{x},\mathbf{p}). \end{aligned} \tag{16}$$

On the other hand we can easily compute them directly (in [17] we already proved that the process $(\mathbf{Y}_n)_{n=0}^{\infty}$ is an homogeneous Markov Chain and we identified the transitions of its components $(Y_n^i)_{n=0}^{\infty}$, $i=1\ldots N$). In fact, if $\mathbf{q}=\mathbf{p}(i,j)$, for some $i\neq j=1\ldots N$, then

$$\begin{aligned} Q(\mathbf{p},\mathbf{p}(i,j)) &= \mathbb{P}[\mathbf{Y}_{n+1}=\mathbf{p}(i,j)|\mathbf{Y}_n=\mathbf{p}] \\ &= \mathbb{P}[\mathbf{Y}_1=\mathbf{p}(i,j)|\mathbf{Y}_0=\mathbf{p}] = \mathbb{P}[Z^0_{\lambda_1-1}=i,\zeta_1=j|\mathbf{Y}_0=\mathbf{p}] \\ &= \mathbb{P}[\zeta_1=j|Z^0_{\lambda_1-1}=i]\cdot\mathbb{P}[Z^0_{\lambda_1-1}=i|\mathbf{Y}_0=\mathbf{p}] = \frac{1}{N-1}\cdot a^i(\mathbf{p}), \end{aligned} \tag{17}$$

where

$$a^i(\mathbf{p}) := \mathbb{P}[Z^0_{\lambda_1-1}=i,|\mathbf{Y}_0=\mathbf{p}] = \sum_{\mathbf{x}\in H} a^i(\mathbf{x},\mathbf{p}). \tag{18}$$

4 Results and conjectures

In [17] we proved a result on the convergence of the associated Markov Chain $(\mathbf{Y}_n)_{n=0}^{\infty}$ to an optimal set of probabilities. This guarantees that, even if we start with a bad set of probabilities, the algorithm finds, in a reasonable number of iterations, the set which decompresses the image in the fastest way.

The convergence results in [17] can be summarized as follows:

Definition 4.1. *The set of probabilities* $\mathbf{p}^* = (p_1^*, \ldots, p_N^*)$ *is* **optimal** *for the corresponding Chaos-Game if*

$$a^i(\mathbf{p}^*) = \frac{1}{N}, \forall i = 1 \ldots N, \tag{19}$$

where a^i *is defined by (18).*

The motivation of this definition comes from the fact that if every map makes mistakes with the same frequence of the others, then the corresponding Chaos-Game will be able to reconstruct the image A as fast as possible. In fact, in such a case, it produces approximately a uniform frequency of visits in the various points of the image.

Theorem 4.2. *At least one optimal set of probabilities for a Chaos Game exists.*

Proof (outline). The proof is based on the fact that $\mathbf{a} = (a^1, \ldots, a^N)$ is a continuous function from a simply connected subset of $\mathbb{R}^N$ into itself. In fact it can be written as an homogeneous polynomial with maximum degree $N_{\mathcal{X}}$ (see (13) and (18)). Continuity turns to be enough to conclude that a point $\mathbf{p}^*$ satisfying (19) exists (proved by induction on N). $\square$

Definition 4.3. *We say that SAOP* **converges almost surely to an optimal set of probabilities** $\mathbf{p}^* = (p_1^*, \ldots, p_N^*)$ *if* $\forall \alpha > 0$, $\exists \bar{n} \in \mathbb{N}$, *s.t.*

$$\lim_{\Delta \to 0} \left(1 - \mathbb{P}\left[\bigcap_{\bar{n} \le n \le k} \{\mathbf{Y}_n \in B_N^{\alpha}(\mathbf{p}^*)\} \right] \right) = 0 \tag{20}$$

with the same order as Δ, *where*

$$B_N^{\alpha}(\mathbf{p}^*) = [p_1^* - \alpha, p_1^* + \alpha] \times \cdots \times [p_N^* - \alpha, p_N^* + \alpha]$$

and k *is any integer greater than* $\bar{n}$.

Theorem 4.4. *If the optimal set of probabilities belongs to* E^N *and is strictly unique, then SAOP converges almost surely, in the sense of the previous definition.*

Proof (outline). The proof is based on the fact that $(Y^i_{n\wedge R^i})_{n=0}^{\infty}$, $i = 1 \dots N$, are super or submartingales (it depends on the starting point), where R^i is the stopping time $R^i = \inf\{n > 0 : a^i(\mathbf{Y}_n) = \frac{1}{N}\}$. Properly using martingale inequalities and the strong Markov property of $(\mathbf{Y}_n)_{n=0}^{\infty}$, we get the thesis. □

In [17], we also conjectured the uniqueness of the optimal set of probabilities, after having performed several simulations with a different number of maps.

It could be interesting to face this problem in connection with the irreducibility property for the associated Markov Chain in the limit of Δ going to zero.

5 The Inverse Problem and SAOP for grey levels

Let us now consider an image represented by a grey level function ℓ on the pixels space $\mathcal{X}$, i.e.

$$\ell : \mathcal{X} \to E = \{0, \Delta, \dots , (M-1)\Delta, 1\} \subset [0, 1]$$

where $\ell(x)$ is the grey level of the pixel x, and, in particular, $\ell(x) = 1$ and $\ell(x) = 0$ correspond to black and white.

Suppose a given set Ξ of N contraction maps is available; then solving the inverse problem, for the image represented by ℓ (and the given set Ξ), consists in finding a set of probabilities $(p_1, \dots , p_N)$ (most of them could in general be equal to zero) such that the frequency of the visits in the various pixels by the associated Chaos-Game approximates ℓ as well as possible. This is equivalent to selecting the "suitable" contraction maps (with the associated probabilities) among the available ones. (For the literature about the inverse problem in image compression see, for example, [1], [8], [9], [10], [15], [20], [21].)

To do this, we modify SAOP, penalizing the "bad map" of a quantity which now depends on the pixel where the mistake occurs: if the pixel is black, we do not care how many times it is visited and the map is not penalized at all; on the other hand, if the pixel is white, the map is drastically penalized and temporarily eliminated.

Let us fix the N maps $\mathbf{w} = (w_1, \dots , w_N)$ and the corresponding probabilities $\mathbf{p} = (p_1, \dots p_N)$. We construct the new Time Dependent Chaos-Game following the original structure of SAOP, but with a different evolution for the probabilities.

Suppose the length of Block j is λ_j as in the classical SAOP: the set of probabilities $\mathbf{p}^j = (p_1^j, \ldots, p_N^j)$ is now

$$p_i^j = \begin{cases} p_i^{j-1} - [(1-\ell(x)) \wedge p_i^{j-1}] & \text{if } Z^{j-1}_{\lambda_j - 1} = i, \\ p_i^{j-1} & \text{if } Z^{j-1}_{\lambda_j - 1} \neq i, \zeta_j \neq i, \\ p_i^{j-1} + [(1-\ell(x)) \wedge p_k^{j-1}] & \text{if } Z^{j-1}_{\lambda_j - 1} = k, \zeta_j = i, \text{ for some } k \neq i \end{cases} \tag{21}$$

where $x = X^j_{\lambda_j}$ is the pixel in which the mistake occurs (i.e. the pixel visited for the second time).

We can consider also the new version of SAOP as an example of GRSCC.

Repeating the arguments used in Section 3, we get the same measurable spaces $(V, \mathcal{V})$ and $(H, \mathcal{H})$. Also the transition probability function $P(\mathbf{p}, A)$, $\forall \mathbf{p} \in V$, $A \in \mathcal{H}$, does not change.

The difference enters in the expression of $\Pi(\mathbf{p}, \mathbf{x}, B)$, $\forall \mathbf{p} \in V$, $\mathbf{x} \in H$, $B \in \mathcal{V}$. In fact, in this case, we have

$$\Pi(\mathbf{p}, \mathbf{x}^{m,l}, \mathbf{q}) = \mathbb{P}[\mathbf{Y}_{n+1} = \mathbf{q} | \mathbf{X}_{n+1} = \mathbf{x}^{m,l}, \mathbf{Y}_n = \mathbf{p}] \neq 0$$

only if $\mathbf{q} = \mathbf{p}(i, j, y)$, for some $i \neq j \in \{1 \ldots N\}$, where

$$p_k(i, j, y) = \begin{cases} p_k & \text{if} \quad k \neq i, j \\ p_i - y & \text{if} \quad k = i \\ p_j + y & \text{if} \quad k = j, \end{cases}$$

and only for the particular value $y = (1 - \ell(x_l^{m,l})) \wedge p_i$.

Then, if $\mathbf{q} = \mathbf{p}(i, j, y)$, for some $i \neq j = 1 \ldots N$, $\forall y \in E$,

$$\begin{aligned} \Pi(\mathbf{p}, \mathbf{x}^{m,l}, \mathbf{p}(i, j, y)) &= \mathbb{P}[\mathbf{Y}_1 = \mathbf{p}(i, j, y) | \mathbf{X}_1 = \mathbf{x}^{m,l}, \mathbf{Y}_0 = \mathbf{p}] = \\ &\mathbb{P}[Z^0_{\lambda_1 - 1} = i, \zeta_1 = j | \mathbf{X}_1 = \mathbf{x}^{m,l}, \mathbf{Y}_0 = \mathbf{p}] \cdot \mathbf{1}_{\{y = (1 - \ell(x_l^{m,l})) \wedge p_i\}} = \\ &\frac{1}{N-1} \cdot \frac{a^i(\mathbf{x}^{m,l}, \mathbf{p})}{\alpha(\mathbf{x}^{m,l}, \mathbf{p})} \cdot \mathbf{1}_{\{y = (1 - \ell(x_l^{m,l})) \wedge p_i\}}. \end{aligned}$$

Then

$$\Pi(\mathbf{p}, \mathbf{x}^{m,l}, \mathbf{q}) = \begin{cases} \frac{1}{N-1} \cdot \frac{a^i(\mathbf{x}^{m,l}, \mathbf{p})}{\alpha(\mathbf{x}^{m,l}, \mathbf{p})} \cdot \mathbf{1}_{\{y = (1 - \ell(x_l^{m,l})) \wedge p_i\}} & \text{if } \mathbf{q} = \mathbf{p}(i, j, y)\ \forall y \in E, \\ & \text{for some } i \neq j = 1 \ldots N, \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

Also in this case, we can find, by (16), the transition probability function $Q(\mathbf{p},\mathbf{q})$ $\forall \mathbf{p},\mathbf{q}\in V$, corresponding to the associated Markov Chain $(\mathbf{Y}_n)_{n=0}^{\infty}$.
If $\mathbf{q}\neq \mathbf{p}(i,j,y), \forall i\neq j=1\dots N,\ y\in E$, then $Q(\mathbf{p},\mathbf{q})=0$, otherwise

$$\begin{aligned} Q(\mathbf{p},\mathbf{q}) &= \sum_{\mathbf{x}\in H}\Pi(\mathbf{p},\mathbf{x},\mathbf{q})\cdot P(\mathbf{p},\mathbf{x}) \\ &= \sum_{\mathbf{x}\in H}\frac{1}{N-1}\cdot\frac{a^i(\mathbf{x},\mathbf{p})}{\alpha(\mathbf{x},\mathbf{p})}\cdot \mathbf{1}_{\{y=(1-\ell(x_l))\wedge p_i\}}\cdot\alpha(\mathbf{x},\mathbf{p}) \\ &= \frac{1}{N-1}\cdot\sum_{\mathbf{x}\in H}a^i(\mathbf{x},\mathbf{p})\cdot\mathbf{1}_{\{y=(1-\ell(x_l))\wedge p_i\}} \end{aligned}$$

6 Comments and outlook

The algorithms described above are in their simplest form. Indeed various empirical modifications can be introduced in SAOP in order to increase the velocity of convergence [4].

As far as the inverse problem is concerned, the proposed SAOP with grey levels is just a first step. In fact, if one works with a fixed finite set of available maps, it will tipically occur that some part of the image is not, even approximately, reproduced.

Some selection algorithms for the case of an a priori unlimited set of randomly generated affine contraction maps are outlined in [18].

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